

# Toy models of a non-associative quantum mechanics

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Toy models of a non-associative quantum mechanics are presented. The Heisenberg equation of motion is modified using a non-associative commutator. Possible physical applications of a non-associative quantum mechanics are considered. The idea is discussed that a non-associative algebra could be the operator language for the non-perturbative quantum theory. In such approach the non-perturbative quantum theory has observables and unobservables quantities.

## I. INTRODUCTION

In Ref. [1] the attempt was made to obtain a possible generalization of quantum mechanics on any numbers including non-associative numbers: octonions. In Ref. [2], the author applies non-associative algebras to physics. This book covers topics ranging from algebras of observables in quantum mechanics, to angular momentum and octonions, division algebras, triple-linear products and Yang - Baxter equations. The non-associative gauge theoretic reformulation of Einstein's general relativity theory is also discussed. In Ref. [3] one can find the review of mathematical definitions and physical applications for the octonions. The modern applications of the non-associativity in physics are: in Refs. [4], [5] it is shown that the requirement that finite translations be associative leads to Dirac's monopole quantization condition; in Ref's [6] and [7] Dirac's operator and Maxwell's equations are derived in the algebra of split-octonions.

In this paper we attempt to give toy models of a non-associative quantum mechanics using finite dimensional non-associative algebras – octonions or sedenions. In the previous paper [8] we have shown that in a non-associative quantum theory the observables can be presented only by elements of an associative subalgebra of a non-associative algebra of non-perturbative quantum operators. Unfortunately now we can not present any model of nonassociative quantum theory since building of such non-associative infinite dimensional algebra is very complicated mathematical problem. But in this paper we present a toy model of non-associative quantum mechanics. We can do that using an analogy with the standard quantum mechanics with the spin: if the relevant degrees of freedom for us are spin degrees of freedom only (the coordinate dependence of a wave function is not important) then we will have a qubit quantum system. The qubit quantum mechanics is much simpler the standard quantum mechanics with Pauli equation.

In this paper we will show that there exists a finite dimensional non-associative algebra (octonions or sedenions) that has a associative subalgebra (quaternions, biquaternions). The associative subalgebra may have a non-commutative subalgebra (quaternions) and a commutative subalgebra. The quantum states of the non-commutative subalgebra are qubits. Three eigenvectors of the commutative subalgebra one can identify with three qubit fermion generations.

Why a non-associative quantum theory can be interesting ? The reason is that it could be a candidate for a nonperturbative quantum theory formulated on the operator language. Generally speaking (according to Ref. [8]) the elements of such algebra are unobservables but if in such non-associative algebra exists an associative subalgebra then their elements are observables.

Thus the goal of this paper is to show that one can find a non-associative finite dimensional algebra having an associative subalgebra (which elements are observables only) and to show that one can correctly define the Heisenberg equation of motion *using the non-associativity property*.

## II. NON-ASSOCIATIVE QUANTUM DYNAMICS

In this section we would like to present a toy model of quantum mechanics realized on a finite dimensional associative subalgebra (quaternions  $\mathbb{Q}$  or biquaternions  $\mathbb{B}$ ) of a non-associative algebra (octonions  $\mathbb{O}$  or sedenions  $\mathbb{S}$ , respectively).

In the usual associative quantum mechanics, we obtain the time evolution of any operator  $\hat{x}$  from the Heisenberg

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equation of motion for the Hamiltonian  $\hat{H}$

$$\frac{d\hat{x}}{dt} = \mathbb{I} [\hat{H}, \hat{x}] \quad (1)$$

where  $[\hat{x}, \hat{y}] = \hat{x}\hat{y} - \hat{y}\hat{x}$  is the commutator and  $\mathbb{I}^2 = -1$ ; later we will omit  $\hat{\cdot}$ . Among many proposed methods for generalizing or modifying the present framework of quantum mechanics, Nambu suggested [9] to modify the Heisenberg equation of motion (1) into a triple product equation

$$\frac{dx}{dt} = \{h_1, h_2, x\} \quad (2)$$

where  $\{x, y, z\}$  is a triple-linear product, and we use two Hamiltonian operators  $h_{1,2}$ , instead of the customary one Hamiltonian as given in Eq. (1).

Let us define a triple product following to [2]. The three-linear product in a vector space  $V$  can be identified by a linear mapping

$$f : V \otimes V \otimes V \rightarrow V. \quad (3)$$

For any  $x, y, z \in V$ , we assign an element  $w \in V$ , which is linear in each of  $x, y, z$ , and we write  $w = f(x, y, z) = \{x, y, z\}$ . The consistency condition

$$\frac{d}{dt}(xy) = x \frac{dy}{dt} + \frac{dx}{dt} y \quad (4)$$

for Eq. (1) leads to

$$\{h_1, h_2, xy\} = x \{h_1, h_2, y\} + \{h_1, h_2, x\} y \quad (5)$$

where  $h_{1,2}, x, y \in V$  and can be any elements of the algebra  $V$  (now the vector space  $V$  simultaneously is an algebra). In Ref. [8] it is shown that if we would like to introduce physical observables in a non-associative quantum theory then they can be elements of an associative subalgebra only. Therefore in contrast with the definition (2) of quantum dynamic on a non-associative algebra  $V$  we will propose that the observable  $x \in V_1 \subset V$  ( $V_1$  is an associative subalgebra of a non-associative algebra  $V$ ) and  $h_{1,2} \in V_1 \setminus V$  are non-associative elements of the algebra  $V$ .

### A. A non-associative quantum mechanics on quaternions

Bearing in mind that the quaternions algebra is equivalent to a qubit algebra one can apply the results of this section to the dynamics of spin, polarized photon and so on.

Let us introduce a quantum non-associative dynamics on  $\mathbb{Q}$  using full non-associative algebra  $\mathbb{O}$  octonions:  $\mathbb{Q} \subset \mathbb{O}$  (in Appendix A the definitions for all algebras and multiplication table are given). For this we introduce a non-accociative commutator (n/a-commutator) in the following way

$$[i_4, i_{m+4}, b] \equiv i_4 (i_{m+4} b) - (b i_4) i_{m+4}, \quad m = 1, 2, 3 \quad (6)$$

generalizing the usual accociative commutator  $[ab, c] = (ab)c - c(ab) = abc - cab$  in the following **non-associative** way

$$[a, b, c] \equiv a(bc) - (ca)b, \quad a, b \in \mathbb{O}, \quad c \in \mathbb{Q} \setminus \mathbb{O}. \quad (7)$$

One can check that

$$[i_4, i_{m+4}, i_n] = -2\varepsilon_{mnk} i_k, \quad m, n, k = 1, 2, 3. \quad (8)$$

The non-triviality of n/a-commutator  $[i_4, i_{m+4}, b]$  is that

$$i_4 (i_{m+4} b) \neq (i_4 i_{m+4}) b \quad \text{and} \quad (b i_4) i_{m+4} \neq b (i_4 i_{m+4}). \quad (9)$$

The consistency condition (4) for Eq. (1) leads to

$$[H, xy] = x [H, y] + [H, x] y \quad (10)$$

which can be easily proved for the associative algebra. But in our case the consistency condition (4) leads to

$$[i_4, i_{m+4}, bc] = b[i_4, i_{m+4}, c] + [i_4, i_{m+4}, b]c, \quad b, c = i_k \in \mathbb{Q}, \quad m, k = 1, 2, 3 \quad (11)$$

and have to be proved. Direct calculations using the Table I (Appendix A) show that it is correct

$$[i_4, i_{m+4}, i_k i_l] = -[i_4, i_{m+4}, i_l i_k] = i_k [i_4, i_{m+4}, i_l] + [i_4, i_{m+4}, i_k] i_l, \quad m, k, l = 1, 2, 3. \quad (12)$$

The octonion  $i_4$  can be replaced with any another octonion  $n = 5, 6, 7$ . It is necessary to note that the consistency condition (11) will be destroyed if the numbers  $b, c = i_{4,5,6,7} \in \mathbb{O} \setminus \mathbb{Q}$ .

For the physical application let us to introduce the following quantities

$$h_{m+4} = i_{m+4} \sqrt{\frac{I\tilde{h}}{2}}, \quad m = 0, 1, 2, 3, \quad (13)$$

$$\hat{S}_k = i_k \frac{I\tilde{h}}{2}, \quad k = 1, 2, 3 \quad (14)$$

here we introduce a new constant  $\tilde{h}$  as it is not evidently that in a non-associative quantum mechanics the Planck constant will be same; then the n/a-commutator will have the form

$$[h_4, h_{m+4}, \hat{S}_n] = -I\tilde{h}\varepsilon_{mnk}\hat{S}_k, \quad m, n, k = 1, 2, 3 \quad (15)$$

which should be compared with the qubit dynamic (43). Now we can define a non-associative quantum dynamic of the quantity  $\hat{\vec{S}} = s_k \hat{S}_k$  in an external vector field  $\tilde{B}_m$  in the following way

$$\frac{d\hat{\vec{S}}}{dt} = I \left[ h_4, -(\vec{B} \cdot \vec{\mu}), \hat{\vec{S}} \right] \quad (16)$$

where  $\vec{B} = \tilde{B}_m \vec{e}_m$  is an analog of a magnetic field,  $\vec{\mu} = \mu h_{m+4} \vec{e}_m$  is an analog of a magnetic dipole for a non-associative case. Inserting  $\hat{\vec{S}} = s_k \hat{S}_k$  into (16) leads to

$$\dot{s}_k = -\varepsilon_{mnk} \omega_m s_n \quad (17)$$

that describes the rotation of the qubit and  $\omega_m = \mu B_m$  is the angular velocity.

At the end of this section we would like to mention that full quantum mechanics on the basis of quaternions can be constructed (for details see Ref. [20]).

## B. A non-associative quantum mechanics on biquaternions

The construction similar to the subsection II A can be done for the biquaternions. In this case

$$[i_4, \epsilon_{m+4}, b] \equiv i_4 (\epsilon_{m+4} b) - (b i_4) \epsilon_{m+4}, \quad m = 1, 2, 3 \quad (18)$$

One can check that

$$[i_4, \epsilon_{m+4}, i_n] = -2\varepsilon_{mnk} \epsilon_k, \quad (19)$$

$$[i_4, \epsilon_{m+4}, \epsilon_n] = 2\varepsilon_{mnk} i_k, \quad m, n, k = 1, 2, 3. \quad (20)$$

The non-triviality of n/a-commutator  $[i_4, \epsilon_{m+4}, b]$  is that

$$i_4 (\epsilon_{m+4} b) \neq (i_4 \epsilon_{m+4}) b \quad \text{and} \quad (b i_4) \epsilon_{m+4} \neq b (i_4 \epsilon_{m+4}). \quad (21)$$

In this case the consistency condition (4) leads to

$$[i_4, \epsilon_{m+4}, bc] = b[i_4, \epsilon_{m+4}, c] + [i_4, \epsilon_{m+4}, b]c, \quad b, c = i_k, \epsilon_k \in \mathbb{B}, \quad m, k = 1, 2, 3 \quad (22)$$

and have to be proved. Direct calculations using the Table I show that it is correct

$$[i_4, \epsilon_{m+4}, i_k i_l] = -[i_4, \epsilon_{m+4}, i_l i_k] = i_k [i_4, \epsilon_{m+4}, i_l] + [i_4, \epsilon_{m+4}, i_k] i_l, \quad (23)$$

$$[i_4, \epsilon_{m+4}, i_k \epsilon_l] = -[i_4, \epsilon_{m+4}, \epsilon_l i_k] = i_k [i_4, \epsilon_{m+4}, \epsilon_l] + [i_4, \epsilon_{m+4}, i_k] \epsilon_l, \quad (24)$$

$$[i_4, \epsilon_{m+4}, \epsilon_k i_l] = -[i_4, \epsilon_{m+4}, i_l \epsilon_k] = \epsilon_k [i_4, \epsilon_{m+4}, i_l] + [i_4, \epsilon_{m+4}, \epsilon_k] i_l, \quad (25)$$

$$[i_4, \epsilon_{m+4}, \epsilon_k \epsilon_l] = -[i_4, \epsilon_{m+4}, \epsilon_l \epsilon_k] = \epsilon_k [i_4, \epsilon_{m+4}, \epsilon_l] + [i_4, \epsilon_{m+4}, \epsilon_k] \epsilon_l, \quad m, k, l = 1, 2, 3. \quad (26)$$

The octonion  $i_4$  can be replaced with any another octonion  $i_{n+4}$  with  $n = 1, 2, 3$ . It is necessary to note that the consistency condition (22) will be destroyed if the numbers  $b, c = i_{k+4}, \epsilon_{k+4}, k = 1, 2, 3$  belong to  $\mathbb{S} \setminus \mathbb{B}$ .

For the physical application let us to introduce the following quantities

$$h_{m+4} = \epsilon_{m+4} \sqrt{\frac{I\hbar}{2}}, \quad m = 0, 1, 2, 3; \quad (27)$$

$$\hat{S}_k = i_k \frac{I\hbar}{2}, \quad k = 1, 2, 3; \quad (28)$$

$$\hat{L}_k = \epsilon_k \frac{I\hbar}{2}, \quad k = 1, 2, 3 \quad (29)$$

then the n/a-commutator will have the form

$$[h_4, h_{m+4}, \hat{S}_n] = -I\hbar \varepsilon_{mnk} \hat{L}_k, \quad m, n, k = 1, 2, 3; \quad (30)$$

$$[h_4, h_{m+4}, \hat{L}_n] = I\hbar \varepsilon_{mnk} \hat{S}_k, \quad m, n, k = 1, 2, 3. \quad (31)$$

Now we can define a non-associative quantum dynamic of the quantities  $\hat{\vec{S}} = s_k \hat{S}_k, \hat{\vec{L}} = l_k \hat{L}_k$  in an external vector fields  $\vec{B}_{1,2;m}$  in the following way

$$\frac{d\hat{\vec{S}}}{dt} = I \left[ h_4, -n_1 (\vec{B}_1 \cdot \vec{\mu}), \hat{\vec{L}} \right], \quad (32)$$

$$\frac{d\hat{\vec{L}}}{dt} = -I \left[ h_4, -n_2 (\vec{B}_2 \cdot \vec{\mu}), \hat{\vec{S}} \right] \quad (33)$$

where  $n_{1,2} = \pm 1$  and describe the sign of the interaction of the fields  $\vec{B}_{1,2}$  with  $\vec{\mu}_{1,2}$  and with the same definitions  $\vec{B}_{1,2} = \vec{B}_{1,2;m} \vec{e}_m$  and  $\vec{\mu} = \mu h_{m+4} \vec{e}_m$  as in the previous section. Inserting  $\hat{\vec{S}} = s_k \hat{S}_k$  and  $\hat{\vec{L}} = l_k \hat{L}_k$  into (16) leads to

$$\dot{s}_k = -\varepsilon_{mnk} \omega_m l_n, \quad (34)$$

$$\dot{l}_k = \varepsilon_{mnk} \omega_m s_n, \quad (35)$$

with the same definition of  $\omega_{1,2;m} = \mu \vec{B}_{1,2;m}$  as the angular velocity.

### III. QUBITS QUANTUM MECHANICS

In this section we would like to present qubit system where above mentioned quantum operators from an associative subalgebra of a non-associative algebra could be operate. Here we follow to the textbook [10]. A qubit is a quantum - mechanical two-state system. A canonical example of a qubit is provided by the spin of a spin-1/2 particle, polarized photon and so on.

Many investigations of quantum systems do not require a “complete” description of the state. For example, one often neglects the position and momentum of a particle when one is only interested in the “inner degrees of freedom” related to the spin. This simplifies the description considerably, because the Hilbert space describing the spin of a particle with spin 1/2 is just the two-dimensional complex vector space  $\mathbb{C}_2$ .

**Definition:** A quantum system with a two-dimensional Hilbert space is called a two-state system or a qubit (quantum bit). The vectors in the Hilbert space of a qubit are often called spinors.

With respect to this basis, vectors are represented by column vectors  $\mathbb{C}_2$ , and linear operators are represented by two-by-two matrices. For example, the basis vectors become

$$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (36)$$

A general state of a qubit is an arbitrary superposition of the two basis states,

$$\psi = c_+ \psi_+ + c_- \psi_- = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \quad \text{with } c_{\pm} \in \mathbb{C} \quad (37)$$

The norm of  $\psi$  and the scalar product with  $\phi = d_+ \psi_+ + d_- \psi_-$  are given by

$$\|\psi\|^2 = |c_+|^2 + |c_-|^2, \quad \langle \psi, \phi \rangle = c_+^* d_+ + c_-^* d_- \quad (38)$$

Any observable has to be represented by a self-adjoint operator. With respect to a chosen orthonormal basis in the Hilbert space of a qubit, observables are thus represented by Hermitian two-by-two matrices: the three Pauli matrices  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  which are the standard representation of the spin observables  $S_1, S_2, S_3$  and as well are the representation of quaternions.

The three Pauli matrices together with the two-dimensional unit matrix  $\mathbf{1}_2$  form a basis in the four-dimensional real vector space of all Hermitian two-by-two matrices. With respect to an orthonormal basis in  $\mathbb{C}^2$ , any qubit observable  $Q$  is represented by a linear combination of Pauli matrices

$$Q = \frac{1}{2} \left( a_0 \mathbf{1}_2 + \sum_{k=1}^3 a_k \sigma_k \right) = \frac{1}{2} \begin{pmatrix} a_0 + a_3 & a_1 - i a_2 \\ a_1 + i a_2 & a_0 - a_3 \end{pmatrix} \quad (39)$$

with real coefficients  $a_0, \dots, a_3$ . It is necessary to note the generation relation for the algebra of Pauli matrices

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \quad (40)$$

One can introduce the spin operators  $\hat{S}_i = \frac{\hbar}{2} \sigma_i$  and then Eq. (40) has the form

$$[\hat{S}_i, \hat{S}_j] = i \epsilon_{ijk} \hat{S}_k \quad (41)$$

A general time-independent qubit Hamiltonian has the form

$$H = a_0 \mathbf{1}_2 + \vec{\omega} \vec{\sigma} \quad (42)$$

where  $\vec{\omega} = \frac{e\vec{B}}{mc}$  is the angular velocity speed,  $e$  and  $m$  is the charge and mass of a particle,  $\vec{B}$  is a magnetic field. The dynamic of the spin  $\hat{\vec{S}} = s_i \hat{S}_i$  is described in the following way

$$\frac{d\hat{\vec{S}}}{dt} = \mathbf{I} \left[ H, \hat{\vec{S}} \right] \quad (43)$$

or

$$\dot{s}_k = \epsilon_{ijk} \omega_i s_j \quad (44)$$

that describes the rotation of the qubit.

#### IV. POSSIBLE PHYSICAL APPLICATIONS

In the section II we presented a non-associative quantum dynamic. We suppose that the non-associative quantization procedure could be applied of a non-perturbative quantization for a field theory. The non-associative quantum dynamic presented in the section II can be an approximation in this direction. Now we would like to present a few possible physical applications of such non-associative quantum dynamic.

### A. Anomalous qubit rotation

Let us consider Eq. (17) describing the qubit rotation under action of an external constant field  $\vec{B} = (0, 0, \tilde{B})$  (which can be not a magnetic field in our non-perturbative case). Then we have the following equations

$$\dot{s}_x = \mu \tilde{B} s_y, \quad (45)$$

$$\dot{s}_y = -\mu \tilde{B} s_x, \quad (46)$$

with the solution

$$s_x = s_{0x} \sin(\omega t), \quad (47)$$

$$s_y = s_{0y} \cos(\omega t). \quad (48)$$

The solution describes the rotation of qubit around the external constant field  $\vec{B}$  in the plane  $xy$ . Comparing Eq's (17) and (44) we see that the rotation of qubit in the non-associative quantum mechanics is in the opposite direction in comparison with the rotation of spin around the magnetic field in the standard quantum mechanics.

For the extended version of qubit presented in subsection II B we use Eq's (34) (35) with

$$\tilde{B}_{1,z} = \tilde{B}_1, \quad \omega_{1,z} = \omega_1, \quad (49)$$

$$\tilde{B}_{2,z} = \tilde{B}_2, \quad \omega_{2,z} = \omega_2, \quad (50)$$

$$\dot{s}_x = \omega_1 l_y, \quad (51)$$

$$\dot{l}_y = \omega_2 s_x \quad (52)$$

with following solution

$$s_x = s_{0x} e^{\pm t \sqrt{n_1 n_2 \omega_1 \omega_2}}, l_y = l_{0y} e^{\pm t \sqrt{n_1 n_2 \omega_1 \omega_2}}. \quad (53)$$

It seems that the solution with  $n_1 n_2 > 0$  is physically senseless as in this case we have exponentially increasing/decreasing operators  $\hat{L}, \hat{S}$ . But in the case  $n_1 n_2 < 0$  we have the rotation of extended qubits in a plane.

### B. Fermion qubit generations

Let us consider the commutative and associative subalgebra  $\mathbb{A}$  spanned on basis  $(1, i_3, \epsilon_3, i_0)$  which is the commutative subalgebra of the noncommutative and associative algebra of biquaternions  $\mathbb{A} \subset \mathbb{B} \subset \mathbb{S}$  (for the definitions of biquaternions and sedenions see Appendix A). The matrix representation of  $\mathbb{A}$  is

$$i_3 = \mathbf{I} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (54)$$

$$\epsilon_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (55)$$

$$i_0 = \mathbf{I} \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (56)$$

The basis vectors are

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (57)$$

which are eigenvectors of matrixes (54)-(56). For such system we can apply Heisenberg equation of motion (32) (33).

One can say that the index  $k$  (by  $\xi_k$ ) enumerate "a fermion generation" of extended qubits living in the vector space  $E$  of the matrix representation of biquaternions. The vector space  $E_1$  spanned on the basis vectors (57) is a

vector subspace  $E_1 \subset E$ . According to equation of motion (34) (35) the generations of extended qubits can mix up. But it is correct only if there exists the interaction term  $(-\vec{B}_2 \cdot \vec{\mu})$  in the opposite case the qubit generations can not be mix up.

In the standard model of particle physics [11] there are open questions which have not yet found an answer. Chief among these is the fermion family or generation puzzle as to why the first generation of quarks and leptons (up quark, down quark, electron and electron neutrino) are replicated in two families or generations of increasing mass (the second generation consisting of charm quark, strange quark, muon and muon neutrino; the third generation consisting of top quark, bottom quark, tau and tau neutrino). One can presuppose that a non-associative infinite dimensional quantum theory may shed light on the generation puzzle of fermions.

### C. Slave-boson decomposition

The most important for the toy model of a non-associative quantum mechanics presented here is the factorization of Hamiltonian in the sense that instead of usual commutator

$$[H, x] = Hx - xH \quad (58)$$

we use a non-associative commutator

$$[h_4, h_{m+4}, x] = h_4(h_{m+4}x) - (xh_4)h_{m+4}. \quad (59)$$

Roughly speaking one can say that the Hamiltonian  $H$  is factorized on two non-associative factors  $h_4, h_{m+4}$ . One can try to find some connection of such decomposition with something similar in physics. In this connection one can think of “slave-boson decomposition” in the  $t - J$  model of High- $T_c$  superconductivity.

It is widely believed that the low energy physics of High- $T_c$  cuprates is described in terms of  $t$ - $J$  type model, which is given by [12]

$$H = \sum_{i,j} J \left( S_i \cdot S_j - \frac{1}{4} n_i n_j \right) - \sum_{i,j} t_{ij} \left( c_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.} \right) \quad (60)$$

where  $t_{ij} = t, t', t''$  for the nearest, second nearest and 3rd nearest neighbor pairs, respectively. The effect of the strong Coulomb repulsion is represented by the fact that the electron operators  $c_{i\sigma}^\dagger$  and  $c_{i\sigma}$  are the projected ones, where the double occupation is forbidden. This is written as the inequality

$$\sum_{\sigma} c_{i\sigma}^\dagger c_{i\sigma} \leq 1 \quad (61)$$

which is very difficult to handle. A powerful method to treat this constraint is so called the slave-boson method [13, 14]. In this approach the electron operator is represented as

$$c_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i \quad (62)$$

where  $f_{i\sigma}^\dagger, f_{i\sigma}$  are the fermion operators, while  $b_i$  is the slave-boson operator. This representation together with the constraint

$$f_{i\uparrow}^\dagger f_{i\uparrow} + f_{i\downarrow}^\dagger f_{i\downarrow} + b_i^\dagger b_i = 1 \quad (63)$$

reproduces all the algebra of the electron operators. The physical meaning of the operators  $f$  and  $b$  is unclear: do exist these fields or not?

If we compare the factorizations (62) and (59) one presuppose that the operators  $f_{i\sigma}^\dagger, b_i$  are elements of an infinite dimensional non-associative algebra  $\Omega$ . This algebra has an associative subalgebra  $\mathfrak{A} \subset \Omega$  and the operator  $c_{i\sigma}^\dagger \in \mathfrak{A}$  is observable but the operators  $f_{i\sigma}^\dagger, b_i \in \Omega \setminus \mathfrak{A}$  are unobservables. It could mean that the High- $T_c$  superconductivity (similar to quantum chromodynamics) can be understood on the basis of a *non-perturbative* quantum theory and one can assume that the non-perturbative quantum theory (on the operator language) could be realized as a non-associative quantum theory (realized as a non-associative algebra  $\Omega$ ) with observables belonging to an associative subalgebra  $\mathfrak{A}$  and unobservables belonging to  $\Omega \setminus \mathfrak{A}$ .

The difference between the slave-boson decomposition (62) and the non-associative commutator (59) is that the first one is a quantum theory where the coordinate degrees of freedom are taking into account, but the second one is a quantum mechanics where the coordinate degrees of freedom are not taking into account.

It is necessary to note here that in Ref's [15]-[18] there are a classical generalization of slave-boson decomposition on gauge theories, so called – “spin-charge separation”.

## V. OUTLOOK

Thus we have shown that one can generalize the standard finite dimensional quantum mechanics (for example, qubit quantum mechanics) to a non-associative finite dimensional quantum mechanics realized on a finite dimensional associative algebra which is a subalgebra of a non-associative algebra. We have considered two cases: quaternionic and biquaternionic non-associative qubit quantum mechanics. In both cases the non-associativity is realized in the Heisenberg quantum equation of motion: on RHS of corresponding equations the usual commutator is changed on a non-associative commutator.

The biquaternion version of non-associative quantum mechanics has commutative  $(1, i_0, \epsilon_3, i_3)$  and noncommutative  $(1, i_1, i_2, i_3)$  observables. It allow us to suppose that an infinite dimensional non-associative quantum theory will have an associative subalgebra having physical observables with commutators  $(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = I\hbar)$  and anticommutators  $(\hat{f}\hat{f}^\dagger + \hat{f}^\dagger\hat{f} = I\hbar)$ . Probably it means that the unification of bosons and fermions can be done not only on the basis of supersymmetry but in a non-associative quantum theory as well.

Now we would like to list the results and properties of discussed here a non-associative quantum mechanics:

- Two examples of a finite dimensional quantum mechanics are presented.
- The Heisenberg quantum equation of motion are essentially non-associative.
- The non-associativity leads to the fact that the usual Hamiltonian can not be introduced as the product of two operators.
- Generally speaking the non-associative factors  $(i_4, i_{m+4})$  or  $(i_4, \epsilon_{m+4})$  are unobservable physical quantities that remind hidden parameters in the theory with hidden parameters.
- The non-associative quantum theory can be alternative one to supersymmetric theories.
- In Ref. [8] it is shown that the non-associative quantum theory can describe non-local objects like strings and so on.

## APPENDIX A: SEDENIONS

Sedenions [19] form an algebra with non-associative but alternative multiplication and a multiplicative modulus. It consists of one real axis (to basis 1), eight imaginary axes (to bases  $i_n$  with  $i_n^2 = -1, n = 0, \dots, 7$ ), and seven real axes (to bases  $\epsilon_n$  with  $\epsilon_n^2 = +1, n = 1, \dots, 7$ ). The multiplication table is given in Table I. The sedenions non-associative algebra contains following important subalgebras:

- the associative quaternion subalgebra  $\mathbb{Q}$  with  $i_n, n = 1, 2, 3$ ;
- the associative biquaternion subalgebra  $\mathbb{B}$  with  $i_0, i_n, \epsilon_n, n = 1, 2, 3$ ;
- the non-associative octonion subalgebra  $\mathbb{O}$  with  $i_n, n = 0, \dots, 7$ .

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|              |              |               |               |               |               |               |               |               |               |               |               |               |               |               |               |               |
|--------------|--------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
|              | 1            | $i_1$         | $i_2$         | $i_3$         | $i_4$         | $i_5$         | $i_6$         | $i_7$         | $i_0$         | $\epsilon_1$  | $\epsilon_2$  | $\epsilon_3$  | $\epsilon_4$  | $\epsilon_5$  | $\epsilon_6$  | $\epsilon_7$  |
| 1            | 1            | $i_1$         | $i_2$         | $i_3$         | $i_4$         | $i_5$         | $i_6$         | $i_7$         | $i_0$         | $\epsilon_1$  | $\epsilon_2$  | $\epsilon_3$  | $\epsilon_4$  | $\epsilon_5$  | $\epsilon_6$  | $\epsilon_7$  |
| $i_1$        | $i_1$        | -1            | $i_3$         | $-i_2$        | $i_5$         | $-i_4$        | $-i_7$        | $i_6$         | $-\epsilon_1$ | $i_0$         | $\epsilon_3$  | $-\epsilon_2$ | $\epsilon_5$  | $-\epsilon_4$ | $-\epsilon_7$ | $\epsilon_6$  |
| $i_2$        | $i_2$        | $-i_3$        | -1            | $i_1$         | $i_6$         | $i_7$         | $-i_4$        | $-i_5$        | $-\epsilon_2$ | $-\epsilon_3$ | $i_0$         | $\epsilon_1$  | $\epsilon_6$  | $\epsilon_7$  | $-\epsilon_4$ | $-\epsilon_5$ |
| $i_3$        | $i_3$        | $i_2$         | $-i_1$        | -1            | $i_7$         | $-i_6$        | $i_5$         | $-i_4$        | $-\epsilon_3$ | $\epsilon_2$  | $-\epsilon_1$ | $i_0$         | $\epsilon_7$  | $-\epsilon_6$ | $\epsilon_5$  | $-\epsilon_4$ |
| $i_4$        | $i_4$        | $-i_5$        | $-i_6$        | $-i_7$        | -1            | $i_1$         | $i_2$         | $i_3$         | $-\epsilon_4$ | $-\epsilon_5$ | $-\epsilon_6$ | $-\epsilon_7$ | $i_0$         | $\epsilon_1$  | $\epsilon_2$  | $\epsilon_3$  |
| $i_5$        | $i_5$        | $i_4$         | $-i_7$        | $i_6$         | $-i_1$        | -1            | $-i_3$        | $i_2$         | $-\epsilon_5$ | $\epsilon_4$  | $-\epsilon_7$ | $\epsilon_6$  | $-\epsilon_1$ | $i_0$         | $-\epsilon_3$ | $\epsilon_2$  |
| $i_6$        | $i_6$        | $i_7$         | $i_4$         | $-i_5$        | $-i_2$        | $i_3$         | -1            | $-i_1$        | $-\epsilon_6$ | $\epsilon_7$  | $\epsilon_4$  | $-\epsilon_5$ | $-\epsilon_2$ | $\epsilon_3$  | $i_0$         | $-\epsilon_1$ |
| $i_7$        | $i_7$        | $-i_6$        | $i_5$         | $i_4$         | $-i_3$        | $-i_2$        | $i_1$         | -1            | $-\epsilon_7$ | $-\epsilon_6$ | $\epsilon_5$  | $\epsilon_4$  | $-\epsilon_3$ | $-\epsilon_2$ | $\epsilon_1$  | $i_0$         |
| $i_0$        | $i_0$        | $-\epsilon_1$ | $-\epsilon_2$ | $-\epsilon_3$ | $-\epsilon_4$ | $-\epsilon_5$ | $-\epsilon_6$ | $-\epsilon_7$ | -1            | $i_1$         | $i_2$         | $i_3$         | $i_4$         | $i_5$         | $i_6$         | $i_7$         |
| $\epsilon_1$ | $\epsilon_1$ | $i_0$         | $\epsilon_3$  | $-\epsilon_2$ | $\epsilon_5$  | $-\epsilon_4$ | $-\epsilon_7$ | $\epsilon_6$  | $i_1$         | 1             | $-i_3$        | $i_2$         | $-i_5$        | $i_4$         | $i_7$         | $-i_6$        |
| $\epsilon_2$ | $\epsilon_2$ | $-\epsilon_3$ | $i_0$         | $\epsilon_1$  | $\epsilon_6$  | $\epsilon_7$  | $-\epsilon_4$ | $-\epsilon_5$ | $i_2$         | $i_3$         | 1             | $-i_1$        | $-i_6$        | $-i_7$        | $i_4$         | $i_5$         |
| $\epsilon_3$ | $\epsilon_3$ | $\epsilon_2$  | $-\epsilon_1$ | $i_0$         | $\epsilon_7$  | $-\epsilon_6$ | $\epsilon_5$  | $-\epsilon_4$ | $i_3$         | $-i_2$        | $i_1$         | 1             | $-i_7$        | $i_6$         | $-i_5$        | $i_4$         |
| $\epsilon_4$ | $\epsilon_4$ | $-\epsilon_5$ | $-\epsilon_6$ | $-\epsilon_7$ | $i_0$         | $\epsilon_1$  | $\epsilon_2$  | $\epsilon_3$  | $i_4$         | $i_5$         | $i_6$         | $i_7$         | 1             | $-i_1$        | $-i_2$        | $-i_3$        |
| $\epsilon_5$ | $\epsilon_5$ | $\epsilon_4$  | $-\epsilon_7$ | $\epsilon_6$  | $-\epsilon_1$ | $i_0$         | $-\epsilon_3$ | $\epsilon_2$  | $i_5$         | $-i_4$        | $i_7$         | $-i_6$        | $i_1$         | 1             | $i_3$         | $-i_2$        |
| $\epsilon_6$ | $\epsilon_6$ | $\epsilon_7$  | $\epsilon_4$  | $-\epsilon_5$ | $-\epsilon_2$ | $\epsilon_3$  | $i_0$         | $-\epsilon_1$ | $i_6$         | $-i_7$        | $-i_4$        | $i_5$         | $i_2$         | $-i_3$        | 1             | $i_1$         |
| $\epsilon_7$ | $\epsilon_7$ | $-\epsilon_6$ | $\epsilon_5$  | $\epsilon_4$  | $-\epsilon_3$ | $-\epsilon_2$ | $\epsilon_1$  | $i_0$         | $i_7$         | $i_6$         | $-i_5$        | $-i_4$        | $i_3$         | $i_2$         | $-i_1$        | 1             |

TABLE I: The sedenions multiplication table.

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